

## [The Inverse Hyperbolic Function]

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**Definition:-** If  $\sinh x = y$ , then  $x$  is called the inverse hyperbolic sine of  $y$  and denoted by the symbol  $\sinh^{-1}y = x$ .

Similarly, we define  $\cosh^{-1}y$  and  $\tanh^{-1}y$  etc.

- Relation between inverse hyperbolic fn and <sup>inverse</sup> circular function: —

If  $\cosh^{-1}x = y$  then  $x = \cosh y = \cos iy$

$$\Rightarrow iy = \cos^{-1}x \Rightarrow y = -i\cos^{-1}x$$

$$\therefore \cosh^{-1}x = -i\cos^{-1}x$$

Similarly,  $\sinh^{-1}x = -i\sin^{-1}(ix)$

$$\tanh^{-1}x = -i\tan^{-1}(ix)$$

- Values of inverse hyperbolic functions: —

It will be seen that inverse hyperbolic fns are many-valued fns. But if  $x$  be real then  $\sinh^{-1}x$  and  $\tanh^{-1}x$  are single-valued functions and  $\cosh^{-1}x$  is a two-valued function.

- Value of  $\cosh^{-1}x$ : —

Let  $\cosh^{-1}x = y$  so that  $x = \cosh y = \frac{e^y + e^{-y}}{2}$

$$\Rightarrow 2x = e^y + \frac{1}{e^y} \Rightarrow \frac{e^{2y} + 1}{e^y}$$

$$\therefore e^{2y} - (2x)e^y + 1 = 0$$

Solving for  $e^y$ , we get

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

Now, we can write that

$$x - \sqrt{x^2 - 1} = \frac{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})}{x + \sqrt{x^2 - 1}} = \frac{x^2 - x^2 + 1}{x + \sqrt{x^2 - 1}}$$
$$= (x + \sqrt{x^2 - 1})^{-1}$$

$$\therefore e^y = (x + \sqrt{x^2 - 1}) \text{ or } (x + \sqrt{x^2 - 1})^{-1}$$

$$\therefore y = 2n\pi i \pm \log(x + \sqrt{x^2 - 1})$$

which gives the general value of  $\cosh^{-1}x$  and its principal value will be when we put  $n=0$  in R.H.S. of the above relation. And neglecting -ve sign.

Principal value is  $\log(x + \sqrt{x^2 - 1})$  (neglecting -ve sign)

[Note: If  $x$  be real,  $\cosh^{-1}x$  is defined only for values of  $x > 1$  and has then the two values  $\pm \log(x + \sqrt{x^2 - 1})$ .]

Value for  $\sinh^{-1}x$ .

$$\text{Let } \sinh^{-1}x = y \text{ then } x = \sinhy = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow 2x = e^y - \frac{1}{e^y} = \frac{e^{2y} - 1}{e^y}$$

$$\therefore e^{2y} - (2x)e^y - 1 = 0$$

Solving for  $e^y$ , we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

$$\therefore e^y = x + \sqrt{x^2 + 1} \text{ or } x - \sqrt{x^2 + 1}$$

Taking logarithm of both the sides, we get

$$\text{either } y = 2n\pi i + \log(x + \sqrt{x^2 + 1}) \text{ or }$$

$$y = \log(-1) + 2n\pi i - \log(x + \sqrt{x^2 + 1})$$

$$\left[ \text{Note- } \because x - \sqrt{x^2+1} = x - \sqrt{x^2+1} \times \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2+1}} \right]$$

$$= \frac{x^2 - x^2 - 1}{x + \sqrt{x^2+1}} = \frac{-1}{x + \sqrt{x^2+1}} = (-1) \cdot (x + \sqrt{x^2+1})^{-1}$$

$$\therefore y = \log(-1) + \log(x + \sqrt{x^2+1})^{-1} + 2n\pi i \quad [\text{for general value}]$$

$$\text{now, } \log(-1) = \log(\cos\pi + i\sin\pi) = \log(e^{i\pi})$$

$$= \log(e^{i\pi} \cdot 1) = \log(e^{i\pi} \cdot e^{2k\pi i}) = \log(e^{(2k+1)i\pi})$$

$$= (2k+1)\pi i].$$

$$\therefore y = 2n\pi i + \log(x + \sqrt{x^2+1}) - (1) \text{ and}$$

$$y = \left(\frac{2m+1}{2k+1}\right)\pi i - \log(x + \sqrt{x^2+1}) - (2)$$

Both the values of (1) and (2) can be included in the expression

$$y = n\pi i + (-1)^n \log(x + \sqrt{x^2+1})$$

This gives general value of  $\sin^{-1}x$ .

$$\text{Its principal value} = \log(x + \sqrt{x^2+1})$$

Note:- If  $x$  be real then  $\sin^{-1}x$  has single value

$$\log(x + \sqrt{x^2+1}) + x.$$

• Value for  $\tan^{-1}x$ . —

$$\text{Let } \tan^{-1}x = y \Rightarrow x = \tan y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$\therefore x = \frac{e^{2y} - 1}{e^{2y} + 1}$$

By using Componendo and dividendo, we get

$$e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = 2n\pi i + \log\left(\frac{1+x}{1-x}\right)$$

$$\text{i.e. } y = n\pi i + \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$\therefore$  The general value of  $\tanh^{-1}x = n\pi i + \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

and its principal value =  $\underline{\frac{1}{2} \log\left(\frac{1+x}{1-x}\right)}$

[For real value of  $x$ ,  $\tanh^{-1}x$  is defined only for the range of values  $-1 < x < 1$  and has the single value is  $\underline{\frac{1}{2} \log\left(\frac{1+x}{1-x}\right)}$ .]

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